



# Negative certainty independence without betweenness<sup>☆</sup>



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## HIGHLIGHTS

- Preferences satisfying negative certainty independence but violating betweenness.
- Answer an open question in the literature, posed in Dillenberger (2010).
- Extend the scope of applications of the negative certainty independence axiom.

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## ABSTRACT

Dillenberger (2010) introduced the negative certainty independence (NCI) axiom, which captures the certainty effect phenomenon. He left open the question of whether there are continuous and monotone preference relations over simple lotteries that satisfy NCI but do not belong to the betweenness class of preferences considered by Chew (1989) and Dekel (1986). We answer this question in the affirmative.

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## 1. Introduction

Dillenberger (2010) suggests *negative certainty independence* (henceforth NCI) as a behavioral axiom, imposed on preferences over monetary lotteries, that captures Kahneman and Tversky's (1979) idea of "certainty effect". NCI is a weakening of the standard vNM-independence that can accommodate, for example, the typical behavior reported in experimental studies of the Allais paradoxes (common ratio and common consequence effects). Dillenberger explored the implications of NCI in other domains, and established an equivalence between static preferences that satisfy NCI, dynamic preferences that display preferences for

one-shot resolution of uncertainty, and preferences over information structures that rank perfect information as the most valuable information system. The leading example of a known model that satisfies NCI is Gul's (1991) theory of disappointment aversion.<sup>1</sup> All known examples, including Gul's model, belong to the class of *betweenness* preferences, studied in Chew (1989) and Dekel (1986). Dillenberger left open the question of whether there are continuous and monotone preferences that satisfy NCI but not betweenness. Since betweenness preferences represent a thin segment of non-expected utility models, and they have relatively little empirical support (see, for example, Camerer and Ho (1994)), answering this question is important in order to understand the scope of application of the NCI axiom. In this note, we provide an example of such a preference relation.

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<sup>1</sup> Gul's model has one additional parameter,  $\beta \in (-1, \infty)$ , compared to the regular expected utility model. Positive values of  $\beta$  correspond to "disappointment aversion" and negative values correspond to "elation seeking." If  $\beta = 0$ , then the model is reduced to expected utility. Gul's preferences satisfy NCI if and only if  $\beta \geq 0$ , that is, if the decision maker is disappointment averse.

**2. The model and main result**

Consider an interval  $[w, b] = X \subset R$  of monetary prizes. Let  $\mathcal{L}$  be the set of all simple lotteries (probability measures with finite support) over  $X$ . For each  $p, q \in \mathcal{L}$  and  $\alpha \in (0, 1)$ , the mixture  $\alpha p + (1 - \alpha) q \in \mathcal{L}$  is the simple lottery that yields each prize  $x$  with probability  $\alpha p(x) + (1 - \alpha) q(x)$ . We denote by  $\delta_x \in \mathcal{L}$  the degenerate lottery that gives the prize  $x$  with certainty, that is,  $\delta_x(x) = 1$ . Note that for any lottery  $p \in \mathcal{L}$  we have  $p = \sum_{x \in X} p(x) \delta_x$ .

Let  $\succeq$  be a preference relation (complete and transitive binary relation) over  $\mathcal{L}$ . We assume that  $\succeq$  is continuous (with respect to the weak topology) and monotone with respect to first-order stochastic dominance. A function  $V : \mathcal{L} \rightarrow \mathbb{R}$  represents  $\succeq$  if  $p \succeq q \Leftrightarrow V(p) \geq V(q)$ .

The following axiom was introduced and motivated by Dillenberger (2010); it is a weakening of the vNM-independence axiom that takes into account the certainty effect.

**Axiom 1 (Negative Certainty Independence (NCI)).** For all  $p, q, \delta_x \in \mathcal{L}$  and  $\lambda \in [0, 1]$ ,  $p \succeq \delta_x$  implies that  $\lambda p + (1 - \lambda) q \succeq \lambda \delta_x + (1 - \lambda) q$ .

In words, the axiom states that, if the decision-maker weakly prefers a lottery to a degenerate lottery yielding a sure monetary prize, then mixing both lotteries with the same third lottery (using the same weight) should not reverse the preference. Roughly speaking, the idea is that a sure outcome suffers more (or gains less) than any lottery from mixtures that eliminate its certainty appeal. As we pointed out in the introduction, NCI played a key role in the analysis of Dillenberger (2010).<sup>2</sup>

The next axiom, which was first introduced by Chew (1989) and Dekel (1986), is a different weakening of the vNM-independence axiom, and is called betweenness.

**Axiom 2 (Betweenness).** For all  $p, q \in \mathcal{L}$  and  $\alpha \in [0, 1]$ ,  $p \succeq q$  implies that  $p \succeq \alpha p + (1 - \alpha) q \succeq q$ .

Betweenness implies neutrality towards mixtures of two indifferently different lotteries.<sup>3</sup> Dillenberger (2010) left open the question of whether there are continuous and monotone preference relations over  $\mathcal{L}$  that satisfy NCI but not betweenness.<sup>4</sup> The next proposition gives an affirmative answer.

<sup>2</sup> To see how NCI can accommodate the behavior observed in Allais' paradoxes, consider, for example, (a version of) Allais' common ratio effect. Subjects choose between  $A$  and  $B$ , where  $A = \$3000$  and  $B = 0.8$  chance of  $\$4000$  and  $0.2$  chance of  $\$0$ . They also choose between  $C$  and  $D$ , where  $C = 0.25$  chance of  $\$3000$  and  $0.75$  chance of  $\$0$ , and  $D = 0.2$  chance of  $\$4000$  and  $0.8$  chance of  $\$0$ . The majority of subjects choose the pair  $(A, D)$ , in violation of expected utility (note that options  $C$  and  $D$  are obtained by mixing options  $A$  and  $B$ , respectively, with the option that yields  $\$0$  for sure). NCI is consistent with this behavior, since the only pattern of choice which is inconsistent with NCI is the pair  $(B, C)$ ; this pattern, however, is rarely observed.

<sup>3</sup> Therefore, betweenness indifference curves in any probability triangle are straight lines, though, unlike in expected utility, they need not be parallel.

<sup>4</sup> Without continuity and monotonicity, it is relatively easy to construct preferences that satisfy NCI and not betweenness. For example, let  $V(p) = \sum_i U(x_i, p)$  where

$$U(x_i, p) = \begin{cases} v(x_i) & p = \delta_{x_i} \\ u(x_i) & \text{otherwise} \end{cases}$$

and  $v(x) > u(x)$  for all  $x$ . These discontinuous preferences first studied by Schmidt (1998) and further analyzed in, for example, Andreoni and Sprenger (2009). These preferences satisfy NCI:  $V(p) \geq V(\delta_x)$  is equivalent to  $\sum_i u(x_i) p(x_i) \geq v(x)$ , and, since  $v(x) > u(x)$ ,

$$\begin{aligned} V(\alpha p + (1 - \alpha) q) &= \sum_i u(x_i) (\alpha p(x_i) + (1 - \alpha) q(x_i)) \\ &\geq \alpha u(x_i) + (1 - \alpha) \sum_i u(x_i) q(x_i) = V(\alpha \delta_x + (1 - \alpha) q). \end{aligned}$$

**Proposition 1.** There is a continuous and monotone function  $V : \mathcal{L} \rightarrow \mathbb{R}$ , which represents preferences that satisfy NCI but not betweenness.

**Proof.** Fix  $n, m$  such that  $w < n < m < b$ . Let

$$u_1(x) = x, \quad \text{and} \\ u_2(x) = \begin{cases} \frac{x+m}{2} & \text{if } x > m \\ x & \text{if } m \geq x \geq n \\ \frac{x+n}{2} & \text{if } n > x. \end{cases}$$

For  $i = 1, 2$ , denote by  $e_i(p)$  the expected utility of a lottery  $p \in \mathcal{L}$  using the function  $u_i$ , that is,  $e_i(p) = \sum_x u_i(x) p(x)$ . Define

$$V(p) = \min \{e_1(p), e_2(p)\}.$$

We now verify that  $V$  satisfies all properties in Proposition 1.<sup>5</sup> Since both  $u_1$  and  $u_2$  are continuous and bounded, their respective expectations are also continuous, and thus  $V$  is continuous. To show monotonicity, note that, if  $p$  first-order stochastically dominates  $q$ , then  $e_i(p) > e_i(q)$  for  $i = 1, 2$ , and thus  $V(p) > V(q)$ , as required.

To show that NCI is satisfied, we first define, for a given lottery  $p$ , the lotteries  $p_L, p_M$ , and  $p_H$  as the restrictions of  $p$  to outcomes, respectively, strictly less than  $n$ , between  $n$  and  $m$ , and strictly larger than  $m$ . Let  $\alpha_L, \alpha_M$ , and  $\alpha_H$  be the probabilities assigned to each interval by  $p$ . Then

$$p = \alpha_L p_L + \alpha_M p_M + \alpha_H p_H.$$

Note that

$$e_1(p) = \alpha_L e_1(p_L) + \alpha_M e_1(p_M) + \alpha_H e_1(p_H)$$

and that

$$\begin{aligned} e_2(p) &= \alpha_L \frac{n + e_1(p_L)}{2} + \alpha_M e_1(p_M) + \alpha_H \frac{m + e_1(p_H)}{2} \\ &:= \frac{\psi(p) + e_1(p)}{2}, \end{aligned}$$

where  $\psi(p) = \alpha_L n + \alpha_M e_1(p_M) + \alpha_H m$ . Since  $e_1(p_M) \in [n, m]$ ,  $\psi(p) \in [n, m]$  as well. Summing up,

$$e_2(p) = \frac{e_1(p) + \psi(p)}{2}, \quad \psi(p) \in [n, m]. \tag{1}$$

Suppose that NCI is not satisfied. Then there exist  $p, q, x$ , and  $\alpha \in (0, 1)$  such that

$$V(p) \geq V(\delta_x), \quad \text{and} \tag{2}$$

$$V(\alpha p + (1 - \alpha) q) < V(\alpha \delta_x + (1 - \alpha) q). \tag{3}$$

Without loss of generality, let  $V(\delta_x) = e_k(\delta_x) \leq e_j(\delta_x)$ , where  $\{j, k\} = \{1, 2\}$ . That is, if  $k = 1$  then  $V(\delta_x) = x$ , and if  $k = 2$  then  $V(\delta_x) = \frac{m+x}{2}$ . From (2), we have

$$e_j(p), e_k(p) \geq \min \{e_j(p), e_k(p)\} = V(p) \geq e_k(\delta_x),$$

These preferences violate betweenness. For example, take  $u(x) = x$  and  $v(x) = 2x$ , and observe that

$$V(\delta_1) = V(0.5\delta_4 + 0.5\delta_0) = 2 > 1.5 = V(0.5\delta_1 + 0.5(0.5\delta_4 + 0.5\delta_0)).$$

<sup>5</sup> When we discuss these properties, we use the function  $V$  and the underlying preferences  $\succeq$  interchangeably.

so

$$e_j(p), e_k(p), e_j(\delta_x) \geq e_k(\delta_x). \quad (4)$$

By (3),

$$\begin{aligned} V(\alpha p + (1 - \alpha)q) &< V(\alpha \delta_x + (1 - \alpha)q) \\ &\leq e_j(\alpha \delta_x + (1 - \alpha)q), e_k(\alpha \delta_x + (1 - \alpha)q). \end{aligned} \quad (5)$$

If  $V(\alpha p + (1 - \alpha)q) = e_k(\alpha p + (1 - \alpha)q)$ , then (5) implies that  $e_k(\alpha p + (1 - \alpha)q) < e_k(\alpha \delta_x + (1 - \alpha)q)$  and, therefore,  $e_k(p) < e_k(\delta_x)$ , which is a contradiction to (4).

If  $V(\alpha p + (1 - \alpha)q) = e_j(\alpha p + (1 - \alpha)q)$ , then (5) implies that  $e_j(\alpha p + (1 - \alpha)q) < e_j(\alpha \delta_x + (1 - \alpha)q)$  and, therefore,

$$e_j(p) < e_j(\delta_x). \quad (6)$$

Thus  $e_k(\delta_x) \leq e_j(p) < e_j(\delta_x)$ , which implies that  $e_k(\delta_x) \neq e_j(\delta_x)$ . Then there are two possibilities: either  $x < n$  or  $x > m$ .

If  $x < n$ , then  $e_1(\delta_x) = x < \frac{x+n}{2} = e_2(\delta_x)$ , so  $k = 1$ . By (4),

$$e_1(p) = e_k(p) \geq e_k(\delta_x) = x.$$

Recall from (1) that  $\psi(p) \geq n$ . Since  $e_1(p) \geq x$ ,

$$e_2(p) = \frac{\psi(p) + e(p)}{2} \geq \frac{n + x}{2} = e_2(\delta_x).$$

Equivalently,  $e_j(p) \geq e_j(\delta_x)$ , which is a contradiction to (6).

If  $x > m$ , then  $e_1(\delta_x) = x > \frac{x+m}{2} = e_2(\delta_x)$ , so  $k = 2$ . By (6),

$$e_1(p) = e_j(p) < e_j(\delta_x) = x.$$

Recall from (1) that  $\psi(p) \geq n$ . Since  $e_1(p) < x$ ,

$$e_2(p) = \frac{\psi(p) + e(p)}{2} < \frac{m + x}{2} = e_2(\delta_x).$$

Equivalently,  $e_k(p) < e_k(\delta_x)$ , which is a contradiction to (4).

Lastly, we show that  $V$  violates betweenness. Pick  $0 < \varepsilon < \min\{2(n - w), b - m, m - n\}$ , and consider the lotteries  $p = \frac{2}{3}\delta_n + \frac{1}{3}\delta_{(m+\varepsilon)}$  and  $q = \frac{1}{3}\delta_{(n-\frac{\varepsilon}{2})} + \frac{1}{3}\delta_{(n+\varepsilon)} + \frac{1}{3}\delta_m$ . Note that

$$e_1(p) = \frac{2}{3}n + \frac{1}{3}m + \frac{1}{3}\varepsilon,$$

$$e_2(p) = \frac{2}{3}n + \frac{1}{3}m + \frac{1}{6}\varepsilon,$$

$$e_1(q) = \frac{2}{3}n + \frac{1}{3}m + \frac{1}{6}\varepsilon,$$

$$e_2(q) = \frac{2}{3}n + \frac{1}{3}m + \frac{1}{4}\varepsilon,$$

so

$$V(p) = e_2(p) = e_1(q) = V(q).$$

For any  $\alpha \in (0, 1)$ ,

$$e_1(\alpha p + (1 - \alpha)q) = \frac{2}{3}n + \frac{1}{3}m + \frac{1}{6}(1 + \alpha)\varepsilon,$$

$$e_2(\alpha p + (1 - \alpha)q) = \frac{2}{3}n + \frac{1}{3}m + \frac{1}{6}\left(\frac{3 - \alpha}{2}\right)\varepsilon,$$

and, therefore,

$$\begin{aligned} V(\alpha p + (1 - \alpha)q) &= \frac{2}{3}n + \frac{1}{3}m + \frac{1}{6}\varepsilon \times \begin{cases} 1 + \alpha & \alpha \leq \frac{1}{3} \\ \frac{3 - \alpha}{2} & \alpha > \frac{1}{3} \end{cases} \\ &\neq V(p) = V(q). \quad \square \end{aligned}$$

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